# Lattice Perturbation Theory from Monte Carlo Simulation


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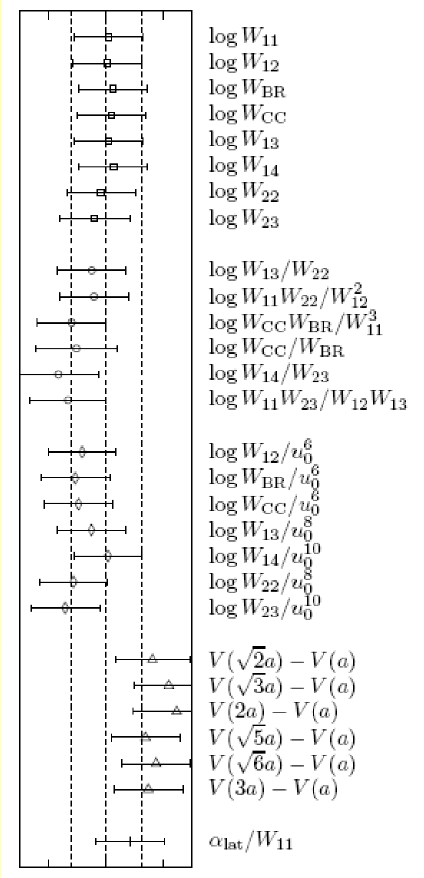
# *Motivation*: we need perturbation theory

example 1. determination of $\alpha_s$ [Q. Mason *et al.*, PRL95, 052002 (2005)]




0.115 0.117 0.119

$\alpha_{\overline{\rm MS}}(M_Z)$

$$\alpha_{\overline{\rm MS}}(M_Z) = 0.1170(12)$$

world average: 0.1187±0.002

- $W_{R,T} \approx c_1 \alpha_V(q^*_{R,T}/a) + c_2 \alpha_V^2(q^*_{R,T}/a) + \ldots$

  measure this in simulation

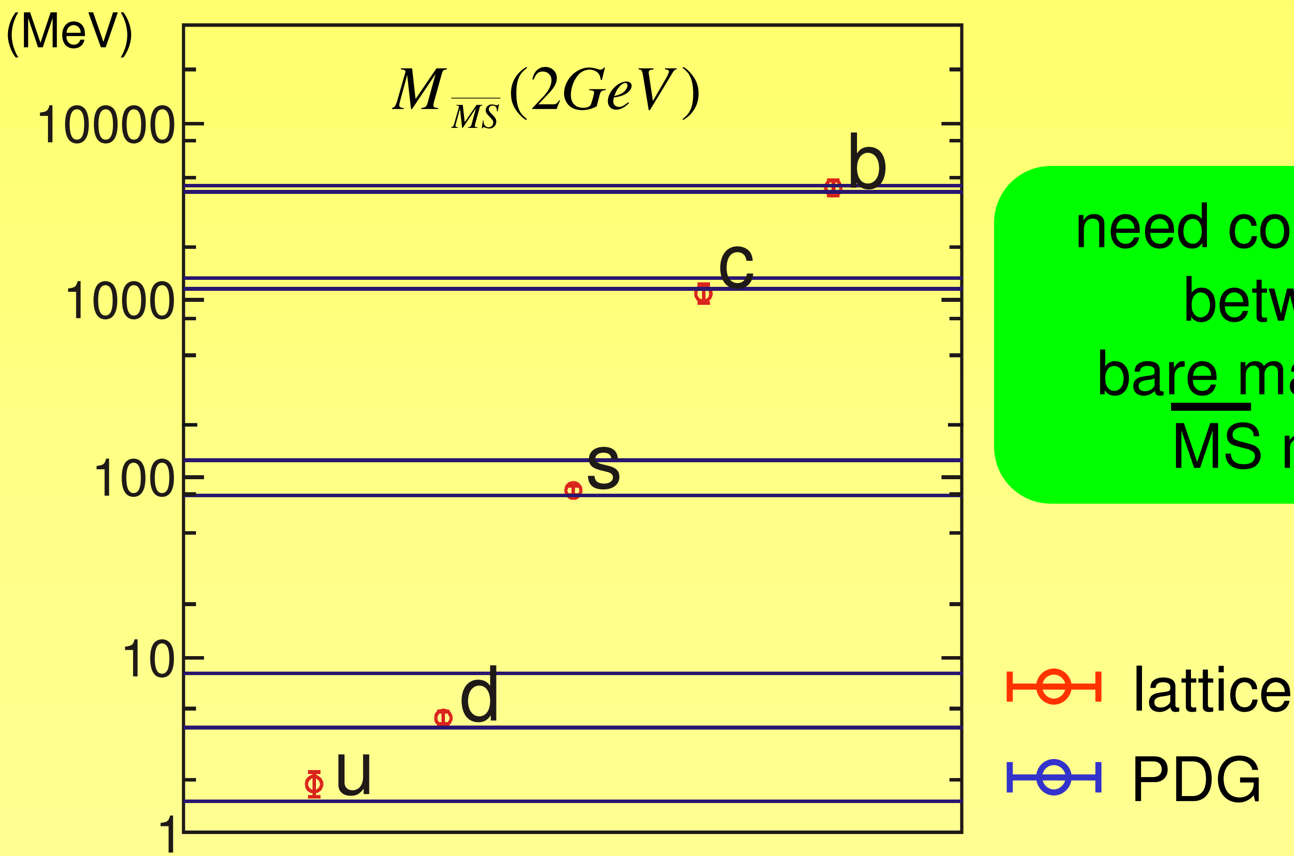
  solve for $\alpha_V(q^*_{R,T}/a)$

- convert $\alpha_V(q^*_{R,T}/a)$ to $\alpha_{\overline{MS}}(M_Z)$

need $c_1$, $c_2$, …

# *Motivation*: we need perturbation theory

example 2. quark masses [Q. Mason *et al.*, hep-ph/0511160]
[M. Nobes & H. Trottier, hep-lat/0509128]

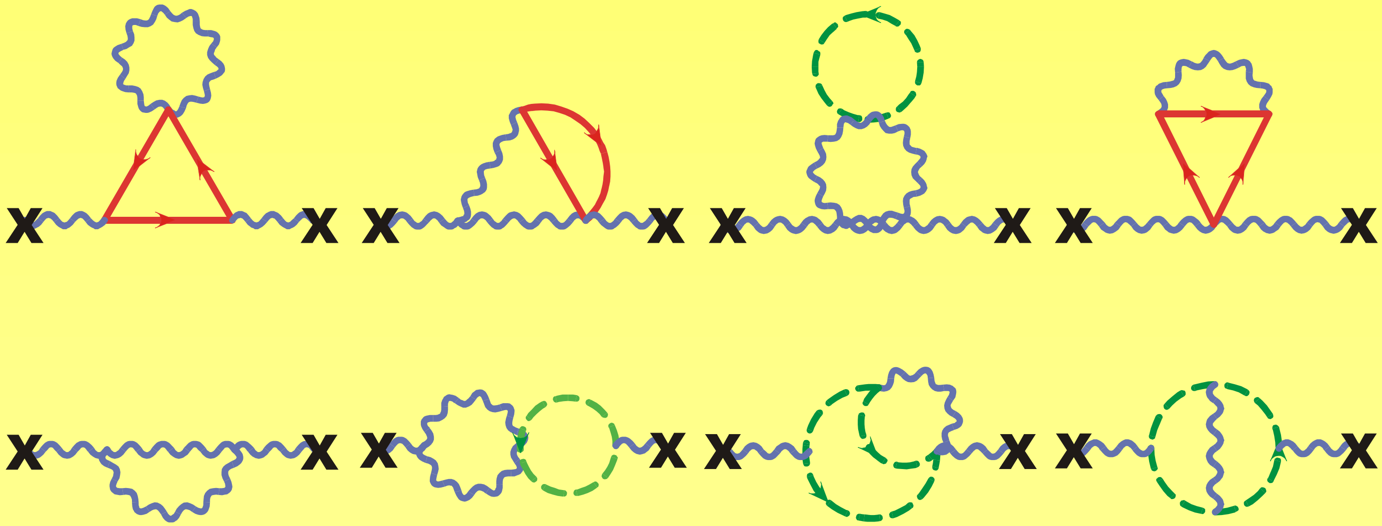



need connection between bare mass and $\overline{\text{MS}}$ mass

# Lattice PT is Difficult!

particularly true for highly improved actions and higher-order calculations

some of the 2-loop diagrams in the expansion for a Wilson loop:



taken from Q. Mason's Ph.D Thesis (Cornell Electronic Library) *so … we need another method*

# Monte Carlo Simulations at Weak Couplings

# Weak Coupling Simulations

*1.* do several simulations at weak couplings

- weak couplings $\rightarrow$ perturbative phase
- weak couplings? $\beta$ ~ 9.5-80.0 ($\alpha_s$ ~ 0.1-0.01)

*2.* fit the data points with an expansion in $\alpha_s$

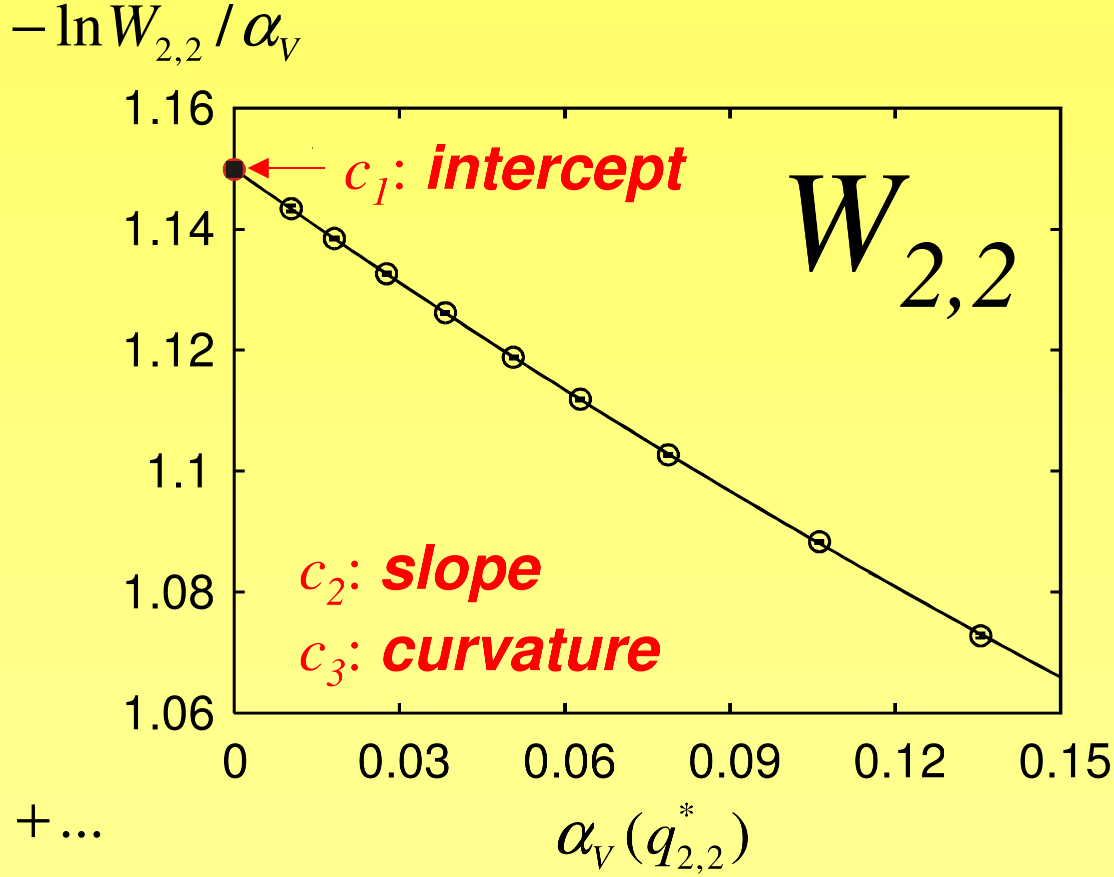
# Weak Coupling Simulations

example

*1.* 9 simulations at weak couplings
→ 9 data points $(\alpha_V, -\ln W_{R,T})$

*2.* fit the points to

$$\frac{-\ln W_{R,T}}{\alpha_V} = c_1 + c_2\alpha_V + c_3\alpha_V^2 + \ldots$$



# Weak Coupling Simulations

*many calculations have been done before:*

- Dimm, Lepage & Mackenzie [Nucl. Phys. B (Proc. Suppl.), **42**, 403 (1995)]

  1st-order mass renormalization for Wilson fermions

- Trottier, Shakespeare, Lepage & Mackenzie [PRD, **65**, 094502 (2002)]

  3rd-order Wilson loops for the Wilson plaquette action

- Hart, Horgan & Storoni [PRD, **70**, 034501 (2004)]

  tadpole factor in pure gauge theories

quenched calculations

***here: unquenched (MILC)*** → important cross-check of the 2-loop expansions used in the $\alpha_s$ calculation

# Difficulties

- need very ***accurate*** measurements:
  statistical errors ~ $10^{-5} – 10^{-6}$ ~ $\alpha_s^3$
- why? want to probe the 3rd-order coefficients
- challenge: ***tight control of systematic errors,*** both in simulations and analysis

  tunneling between $Z_3$ center phases

  fitting and truncation errors

  zero momentum modes

  step size errors in simulation equations

  autocorrelation

  inaccuracy in matrix inversions

# Simulation Parameters

***MILC*** action, 9 couplings

| $\beta$ | $\alpha_0$ | $\alpha_V(q^*_{1,1})$ |
|---|---|---|
| 9.5 | 0.0838 | 0.1278 |
| 11.0 | 0.0723 | 0.1016 |
| 13.5 | 0.0589 | 0.0741 |
| 16.0 | 0.0497 | 0.0619 |
| 19.0 | 0.0419 | 0.0506 |
| 24.0 | 0.0332 | 0.0382 |
| 32.0 | 0.0249 | 0.0271 |
| 47.0 | 0.0169 | 0.0183 |
| 80.0 | 0.0099 | 0.0101 |

- input from analytic PT

$$\alpha_0 \leftrightarrow \alpha_V(q^*_{1,1})$$

$\downarrow$

(not necessary)
use 2-loop expansion
of the plaquette

# Results

***Perturbation Theory*** | ***Monte Carlo Simulation***

| Loop | PT $c_1$ | PT $c_2$ | PT $c_3$ | MC $c_1$ | MC $c_2$ | MC $c_3$ |
|---|---|---|---|---|---|---|
| 1x2 | 0.9252(0) | -0.646(9) | 0.23(5) | 0.9251(3) | -0.644(13) | 0.20(18) |
| 1x3 | 0.9845(0) | -0.595(1) | 0.38(6) | 0.9845(3) | -0.599(14) | 0.37(19) |
| 2x2 | 1.1499(0) | -0.643(2) | 0.59(9) | 1.1499(4) | -0.641(15) | 0.58(20) |
| 2x3 | 1.2341(0) | -0.595(3) | 0.85(16) | 1.2342(4) | -0.599(19) | 0.88(26) |

***Fit with $c_1$ fixed to PT values***

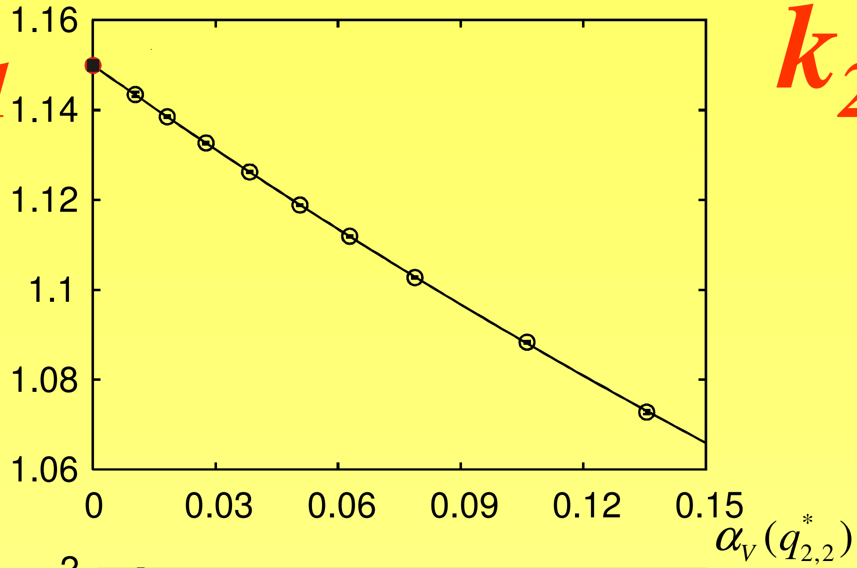
| Loop | $c_2$ | $c_3$ |
|---|---|---|
| 1x2 | -0.649(5) | 0.26(12) |
| 1x3 | -0.600(6) | 0.39(13) |
| 2x2 | -0.642(7) | 0.59(14) |
| 2x3 | -0.594(7) | 0.82(15) |

***Both $c_1$, $c_2$ fixed***

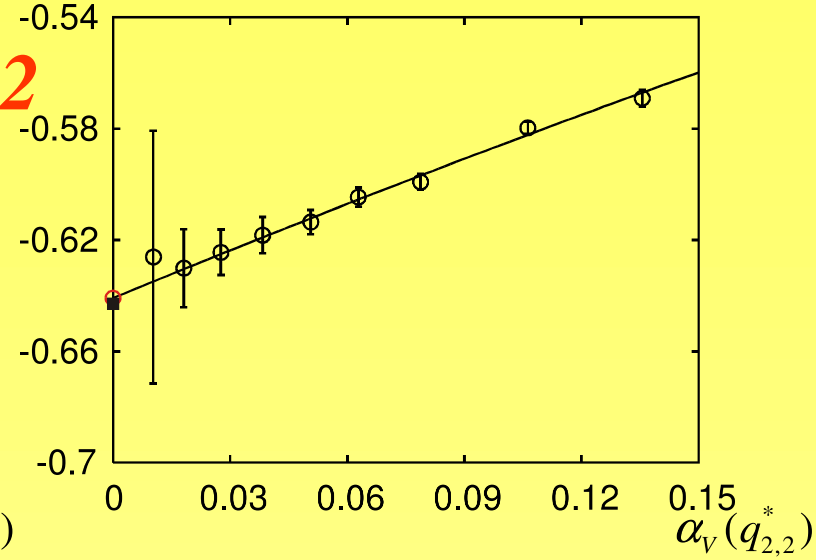
| Loop | $c_3$ |
|---|---|
| 1x2 | 0.21(4) |
| 1x3 | 0.28(5) |
| 2x2 | 0.61(6) |
| 2x3 | 0.84(8) |

# Results – 2x2 Loop

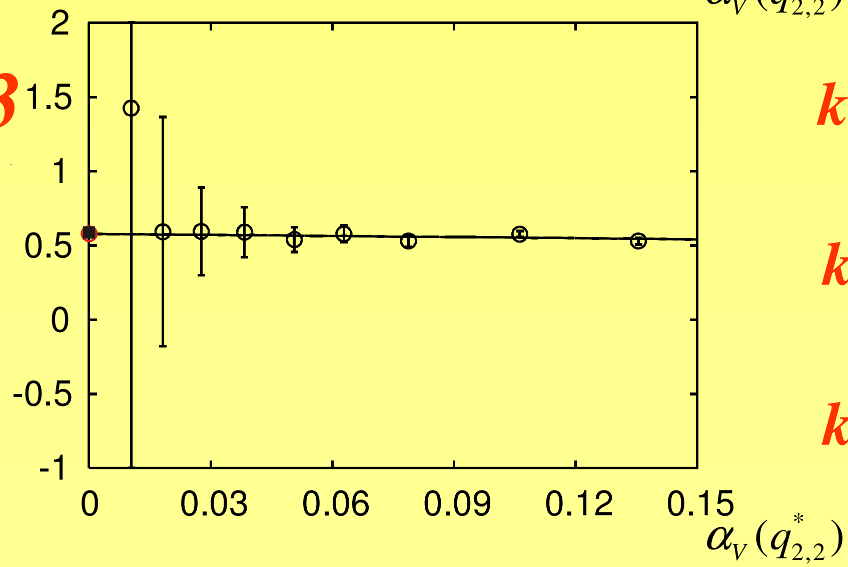
$k_1$




$k_2$




$k_3$




$$k_1 = \frac{-\ln W_{2,2}}{\alpha_V} = c_1 + c_2\alpha_V + \ldots$$

$$k_2 = \frac{-\ln W_{2,2} - c_1\alpha_V}{\alpha_V} = c_2 + c_3\alpha_V \ldots$$

$$k_3 = \frac{-\ln W_{2,2} - c_1\alpha_V - c_2\alpha_V^2}{\alpha_V} = c_3 + \ldots$$

**provides an important cross-check of the 2-loop expansions used in the $\alpha_s$ calculation**

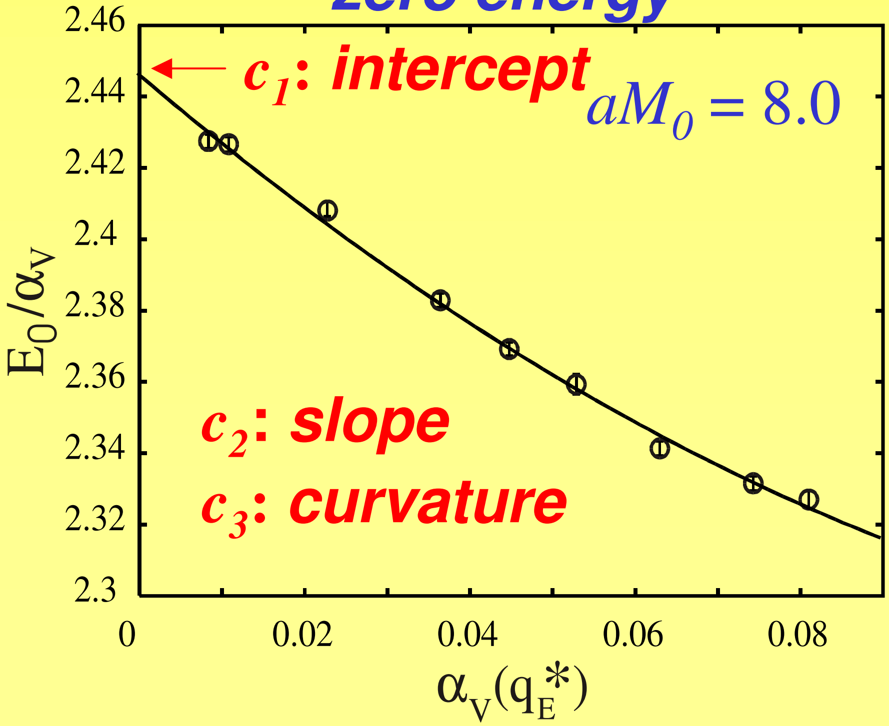
# Results – NRQCD

NRQCD action: $H=-\frac{\Delta^2}{2M_0}$; dispersion relation $E=E_0+\frac{p^2}{2M}$

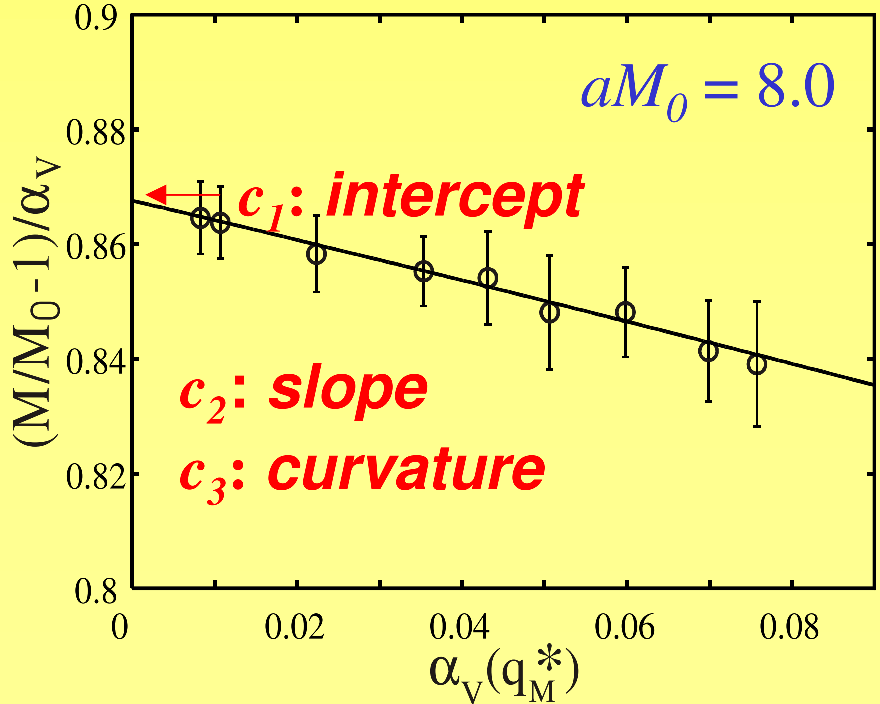
zero energy: $E_0=c_1\alpha_V(q_E^*)+c_2\alpha_V^2(q_E^*)+\ldots$

mass renormalization: $M=M_0\left(1+c_1\alpha_V(q_M^*)+c_2\alpha_V^2(q_M^*)+\ldots\right)$

***zero energy***




***mass renormalization***



# Results – NRQCD

$c_1$: ***zero energy*** ***mass renormalization***

**Zero energy:**

| $M_0a$ | PT | MC |
|---|---|---|
| 8.0 | 2.450 | 2.447(2) |
| 6.0 | 2.570 | 2.565(5) |
| 5.0 | 2.655 | 2.648(5) |
| 4.0 | 2.782 | 2.773(7) |

**Mass renormalization:**

| $M_0a$ | PT | MC |
|---|---|---|
| 8.0 | 0.850 | 0.867(6) |
| 6.0 | 0.830 | 0.849(7) |
| 5.0 | 0.820 | 0.842(9) |
| 4.0 | 0.770 | 0.798(10) |

***Predictions* for $c_2$ and $c_3$:**

| $M_0a$ | $c_2$ | $c_3$ |
|---|---|---|
| 8.0 | -1.98(8) | 5.9(1.4) |
| 6.0 | -1.91(7) | 6.3(1.4) |
| 5.0 | -2.02(7) | 7.3(1.4) |
| 4.0 | -2.13(7) | 7.4(1.4) |

| $M_0a$ | $c_2$ | $c_3$ |
|---|---|---|
| 8.0 | -0.34(37) | -0.2(4.9) |
| 6.0 | -0.33(42) | 0.5(5.3) |
| 5.0 | -0.53(45) | 0.7(6.1) |
| 4.0 | -0.91(56) | -0.28(6.7) |

# Conclusions

*1.* numerical approach is ***simple***; give accurate results

*2.* Wilson loops: impressive agreement

***important for the $\alpha_s$ calculation***

*3.* in progress: HIST, mNRQCD